PARTIAL DIFFERENTIAL EQUATIONS

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3. Linear evolutionary PDE

(1) Prove that, for any $u_\circ \in L^2(\Omega)$, there exists a solution of the Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u &= 0 & \text{in} \quad \Omega \times (0,\infty) \\ u &= 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ u(x,0) &= u_\circ(x) & \text{for} \quad t = 0. \end{cases}$$

and is given by

$$u(x,t) = \sum_{k=1}^{\infty} a_k e^{i\lambda_k t} \phi_k(x), \qquad a_k = \int_\Omega u_\circ \phi_k,$$

where $\{\phi_k\}$ is the sequence of eigenfunctions of the Laplacian in Ω.

(3 points)

(2) Let $u_\circ \in L^2(\Omega)$, and let $u(x,t)$ be the solution of

$$\begin{cases} \partial_t u - \Delta u &= 0 & \text{in} \quad \Omega \times (0,\infty) \\ u &= 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ u(x,0) &= u_\circ(x) & \text{for} \quad t = 0. \end{cases} \tag{0.1}$$

Prove that $u \to u_\circ$ in $L^2(\Omega)$ as $t \to 0$, that is,

$$\lim_{t\to 0} \|u(\cdot,t) - u_\circ\|_{L^2(\Omega)} = 0.$$

(2 points)

(3) Let $u_\circ \in H^m(\Omega)$ for some $m \geq 1$, and let $u(x,t)$ be the solution of (0.1).

(i) Prove that $u \to u_\circ$ in $H^m(\Omega)$ as $t \to 0$, that is,

$$\lim_{t\to 0} \|u(\cdot,t) - u_\circ\|_{H^m(\Omega)} = 0.$$

(ii) Prove that

$$\|u(\cdot,t)\|_{H^m(\Omega)} \leq C\|u_\circ\|_{H^m(\Omega)}$$

for all $t > 0$.

(iii) Prove that

$$\|u(\cdot,t)\|_{H^{m+1}(\Omega)} \leq \frac{C}{t^{1/2}}\|u_\circ\|_{H^m(\Omega)}$$

for all $t > 0$.

(4 points)

(4) (i) Let $u_\circ \in L^2(\Omega)$, and let $u(x,t)$ be the solution of (0.1). Prove the following decay in L^2 norm as $t \to \infty$

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} e^{-\lambda_1 t}$$

for all $t > 0$, where $\lambda_1 > 0$ is the first eigenvalue of the Laplacian in Ω.

(ii) Let $u_\circ \in L^2(\Omega)$, and let $u(x,t)$ be the solution of (0.1). Prove the following decay in L^∞ norm as $t \to \infty$

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C e^{-\lambda_1 t}$$

for all $t \geq 1$, where $\lambda_1 > 0$ is the first eigenvalue of the Laplacian in Ω.

(3 points)

(5) (i) Let u be any solution of the heat equation

$$\partial_t u - \Delta u = 0 \quad \text{in} \quad \Omega \times (0,\infty).$$

Prove that, for any $r > 0$, the function

$$U(x,t) := u(rx, r^2 t)$$

solves the heat equation in the domain $r\Omega := \{rx : x \in \Omega\}$.

How should we rescale the function u in order to obtain a solution of $\partial_t w - \alpha \Delta w = 0$ in $\Omega \times (0,\infty)$, with $\alpha > 0$?

(ii) Let u be any solution of the wave equation

$$\partial_{tt} u - \Delta u = 0 \quad \text{in} \quad \Omega \times (0,\infty).$$

How should we rescale the function u in order to obtain a solution of $\partial_{tt} w - c\Delta w = 0$ in $\Omega \times (0,\infty)$, with $c > 0$?

(2 points)

(6) We have two chickens, one weighting 1kg and the other one 2kg. Initially they are at 20°C, and we want to bake them in the oven at constant temperature, say 200°C.

If it takes 30 minutes to get the first chicken at 100°C everywhere inside, how long will it take for the second chicken?

Note: Assume the two chickens are identical in shape, and find the answer by rescaling.

(3 points)

(7) Let $m > 1$. Assume there exists a positive C^2 solution of $-\Delta S = S^{1/m}$ in Ω, with $S = 0$ on $\partial\Omega$. Use then separation of variables to find an explicit solution of the nonlinear diffusion equation

$$\begin{cases} \partial_t u - \Delta(u^m) &= 0 \quad \text{in} \quad \Omega \times (0,\infty) \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (0,\infty) \end{cases}$$

of the form $u(x,t) = T(t)S^m(x)$, which is C^2 for any $t > 0$, but which satisfies

$$u(x,t) \to +\infty \quad \text{as} \quad t \to 0 \quad \text{for } \textit{every} \quad x \in \Omega.$$

Note: The existence of the function S can be proved by using the methods of Chapter 5.

(3 points)

(8) Prove that there exists a bounded smooth function $f \in C^\infty(\Omega)$ for which there is no solution $u \in C^2(\Omega \times [0,1])$ of

$$\begin{cases} \partial_t u - \Delta u &= 0 \qquad \text{in} \quad \Omega \times (0,1) \\ u(x,1) &= f(x) \qquad \text{for} \quad t = 1. \end{cases}$$

This means that the backwards heat equation is not solvable.

(3 points)

(9) Let u be the solution of the Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u &= 0 \qquad \text{in} \quad \Omega \times (0,\infty) \\ u &= 0 \qquad \text{on} \quad \partial\Omega \times (0,\infty) \\ u(x,0) &= u_\circ(x) \qquad \text{for} \quad t = 0. \end{cases}$$

Prove that, if $u_\circ$ is *not* $C^\infty(\overline{\Omega})$, then u is not $C^\infty(\overline{\Omega})$ for any $t > 0$.

(2 points)

(10) Let $u(x,t)$ be any solution of

$$\begin{cases} \partial_t u - \Delta u &= 0 \qquad \text{in} \quad \Omega \times (0,\infty) \\ u &= 0 \qquad \text{on} \quad \partial\Omega \times (0,\infty). \end{cases}$$

(i) Prove that

$$\frac{d}{dt}\int_\Omega |\nabla u|^2 dx \le 0$$

(ii) Prove that the function

$$J(t) = \log\left(\int_\Omega u^2 dx\right)$$

is convex in t, for all $t > 0$.

(iii) Assume now that we have Neumann boundary conditions $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ (instead of $u = 0$ on $\partial\Omega$) and that $u > 0$ in $\overline{\Omega}$. Prove that

$$\frac{d}{dt}\int_\Omega \log u \, dx \ge 0$$

(3 points)

(11) Let $u(x,t)$ be any solution of

$$\begin{cases} i\partial_t u - \Delta u &= 0 \qquad \text{in} \quad \Omega \times (0,\infty) \\ u &= 0 \qquad \text{on} \quad \partial\Omega \times (0,\infty). \end{cases}$$

Prove that $\int_\Omega |\nabla u|^2 dx$ is constant in time.

(2 points)

(12) Let $u \in C^2(\overline{\Omega} \times [0,\infty))$ be a solution of the wave equation $\partial_{tt} u - \Delta u = 0$ in $\Omega \times [0,\infty)$, with initial data $u(x,0) = u_\circ(x)$ and $u_t(x,0) = v_\circ(x)$, and with *Neumann* boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty).$$

(i) Show that $\int_\Omega \left(|\partial_t u|^2 + |\nabla u|^2\right)$ is constant in time.

(ii) Show that $\int_\Omega \partial_t u$ is constant in time.

(iii) Deduce that $\int_\Omega u(x,t)dx = \int_\Omega u_\circ + t\int_\Omega v_\circ$.

(3 points)

(13) Prove that any function of the form $u(x,t) := v(x \cdot e - t)$, with $|e| = 1$ and $v \in C^2(\mathbb{R})$, is a solution of the wave equation $\partial_{tt}u - \Delta u = 0$ in $\mathbb{R}^n \times (-\infty, \infty)$.

Are there any solutions of the same form for the heat equation or the Schrödinger equation?

(2 points)

(14) Let u be a solution of the Schrödinger equation $i\partial_t u - \Delta u = 0$ in $\mathbb{R}^n \times \mathbb{R}$. Prove that, for any $\xi \in \mathbb{R}^n$, the function

$$v(x,t) := e^{\frac{i}{4}(2\xi\cdot x - |\xi|^2 t)}u(x,t)$$

also solves the Schrödinger equation $i\partial_t v - \Delta v = 0$ in $\mathbb{R}^n \times \mathbb{R}$.

Note: This shows Galilean invariance of solutions.

(2 points)

(15) (i) Prove that the function

$$P(x,t) = 1/(4\pi t)^{n/2} e^{-\frac{|x|^2}{4t}}$$

satisfies the heat equation in $\mathbb{R}^n$ for all $t > 0$.

(ii) Deduce that, for any locally integrable and bounded initial condition $u_\circ$, the function

$$u(x,t) = \int_{\mathbb{R}^n} u_\circ(y)P(x-y,t)dy$$

is $C^\infty(\mathbb{R}^n \times (0,\infty))$ and solves the heat equation in $\mathbb{R}^n$ for all $t > 0$.

(3 points)

(16) Let $b \in C(\mathbb{R}^n \times \mathbb{R})$, and let $u \in C^2(\mathbb{R}^n \times [0,T])$ be a solution of

$$\partial_t u - \Delta u + b \cdot \nabla u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,T)$$

with $u \to 0$ uniformly as $|x| \to \infty$.

Prove that the maximum of u is attained at time $t = 0$, that is,

$$\max_{\mathbb{R}^n \times [0,T]} u(x,t) = \max_{\mathbb{R}^n} u(x,0).$$

(3 points)